Boltzmann Equation for Relativistic Neutral Scalar Field in Non-equilibrium Thermo Field Dynamics

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A relativistic neutral scalar field is investigated on the basis of the Schwinger-Dyson equation in the non-equilibrium thermo field dynamics. A time evolution equation for a distribution function is obtained from a diagonalization condition for the Schwinger-Dyson equation. An explicit expression of the time evolution equation is calculated in the $\lambda\phi^4$ interaction model at the 2-loop level. The Boltzmann equation is derived for the relativistic scalar field. We set a simple initial condition and numerically solve the Boltzmann equation and show the time evolution of the distribution function and the relaxation time.

§1. Introduction

The relativity and the quantum field theory provide a general basis for understanding a model of elementary particles at high energy. The theory should be extended to include a statistical thermodynamics for macroscopic phenomena of many particles systems. Some formalism for thermal quantum field theories is proposed to evaluate in and out of equilibrium systems. The real-time formalism is necessary to include time evolution of the system. Y. Takahashi and H. Umezawa proposed the thermo field dynamics (TFD) which is the real-time formalism based on the canonical quantization.[1)–3)] After that TFD has been applied to various physical systems.

In 1985 T. Arimitsu and H. Umezawa extended TFD to a non-equilibrium system with spatially homogeneous distributions.[4),5)] It has been found that the time development of a particle number density is derived by the self-consistent renormalization condition.[1),6)–8)] An equation with the structure of the Boltzmann equation is derived in the non-equilibrium TFD (NETFD). An alternative procedure to derive a Boltzmann-like equation has been proposed by evaluating the expectation value of the number operator in terms of a renormalized field in Refs.[9),10)] It is also obtained by diagonalizing the Green's function at the equal time limit instead of the thermal self-energy diagonalization scheme in Ref.[11)] TFD for spatially inhomogeneous non-equilibrium system has been studied in Refs.[12)–14)] NETFD successfully describes macroscopic phenomena in a low energy non-relativistic system.

It is also quite interesting to make investigations various phenomena associated with non-equilibrium dynamics at high energy. As far as we know, only a little work has been reported in the study of NETFD for a high energy relativistic system. P. Henning introduced independent weighting functions for relativistic fields with positive and negative energy and described an extended Bogoliubov transformation

for thermal doublets.[14] NETFD was applied for a relativistic scalar field with a four-point self-interaction in Ref.[15] A renormalization condition is imposed on the thermal self-energy. Thus a Boltzmann-like equation is derived from the expression for the particle number density of the scalar field.

In the present paper we would like to find a natural derivation of the self-consistency conditions available for relativistic scalar fields by using the Schwinger-Dyson (SD) equation. In Sec. 2 we briefly review NETFD for a scalar field and introduce the thermal Bogoliubov matrices whose parameter defines a particle number density. In Sec. 3 we evaluate the SD equation in NETFD. A Boltzmann equation is derived from a time evolution of a parameter in the thermal Bogoliubov matrices. In Sec. 4 a scalar field with a $\lambda\phi^4$ interaction is investigated according to the NETFD. We calculate the thermal self-energy at 2-loop level. In Sec. 5 we give an explicit expression for the Boltzmann equation in the $\lambda\phi^4$ interaction model. Solving the Boltzmann equation starting from a simple initial state numerically, we show the time evolution of the particle number density and the relaxation time. We discuss how the off-shell mode contributes the relaxation process. Some concluding remarks are given in Sec. 6.

§2. Non-equilibrium thermo field dynamics for relativistic fields

There are several formalism to introduce the thermal dynamics into the quantum field theory. In TFD the statistical average is replaced by an expectation value in a pure state called the thermal vacuum. The thermal vacuum is defined by extending the Fock space structure. A creation and an annihilation operators, $a^\dagger$, a are doubled by introducing tilde operators, $\tilde{a}^\dagger$, $\tilde{a}$. Thus the commutation relations for the bosonic creation and annihilation operators are extended to be

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \tag{2·1}$$

$$[\tilde{a}_{\boldsymbol{p}}, \tilde{a}^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \tag{2·2}$$

$$\text{others} = 0. \tag{2·3}$$

The time evolution of the non-tilde operators are generated by an ordinary non-thermal Hamiltonian, H, for the considering system. In a similar manner the time evolution for the tilde operators is described by the tilde conjugate Hamiltonian, $\tilde{H}$. In defining the tilde-Hamiltonian, $\tilde{H}$, we use the following tilde conjugation rules,

$$(A_1A_2)^\sim = \tilde{A}_1\tilde{A}_2, \tag{2·4}$$

$$(c_1A_1 + c_2A_2)^\sim = c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2, \tag{2·5}$$

$$(\tilde{A})^\sim = A, \tag{2·6}$$

where c_1 and c_2 are c-numbers, and A_1 and A_2 are arbitrary operators. The tilde-Hamiltonian, $\tilde{H}$, is constructed by only the tilde operators, $\tilde{a}$ and $\tilde{a}^\dagger$. The time evolution of both the non-tilde and tilde operators is described by the total Hamiltonian of the system,

$$\hat{H} \equiv H - \tilde{H}. \tag{2·7}$$

The Fock space is extended to the state space spanned by both the non-tilde and tilde creation operators. The thermal vacuum is defined by the thermal Bogoliubov transformation of the eigenstates of the Hamiltonian. Non-equilibrium degree of freedom can be also introduced in TFD through the thermal Bogoliubov transformation,[1),4),16)]

$$\begin{aligned} \xi_p^\alpha e^{-i\omega_p \cdot t} &= B(n_p(t))^{\alpha\beta} a_p^\beta(t), \\ \bar{\xi}_p^\alpha e^{i\omega_p \cdot t} &= \bar{a}_p^\beta(t) B^{-1}(n_p(t))^{\beta\alpha}, \end{aligned} \tag{2·8}$$

where $\omega_p = \sqrt{\boldsymbol{p}^2 + m^2}$ is the relativistic energy eigenvalue for bosonic fields with a momentum, $\boldsymbol{p}$, and mass, m. The upper indices are defined by the thermal doublets notation,

$$a_p^\alpha = \begin{pmatrix} a_p \\ \tilde{a}_p^\dagger \end{pmatrix}, \quad \bar{a}_p^\alpha = \begin{pmatrix} a_p^\dagger & -\tilde{a}_p \end{pmatrix}, \tag{2·9}$$

$$\xi_p^\alpha = \begin{pmatrix} \xi_p \\ \tilde{\xi}_p^\dagger \end{pmatrix}, \quad \bar{\xi}_p^\alpha = \begin{pmatrix} \xi_p^\dagger & -\tilde{\xi}_p \end{pmatrix}. \tag{2·10}$$

It is assumed that the thermal Bogoliubov matrices, $B(n_p(t))^{\alpha\beta}$ and $B^{-1}(n_p(t))^{\alpha\beta}$, have the same forms with the ones in equilibrium,

$$\begin{aligned} B(n_p(t))^{\alpha\beta} &= \begin{pmatrix} 1 + n_p(t) & -n_p(t) \\ -1 & 1 \end{pmatrix}, \\ B^{-1}(n_p(t))^{\alpha\beta} &= \begin{pmatrix} 1 & n_p(t) \\ 1 & 1 + n_p(t) \end{pmatrix}. \end{aligned} \tag{2·11}$$

The Bogoliubov parameter, $n_p(t)$, depends on time and the relativistic energy eigenvalue through the momentum, $\boldsymbol{p}$.

In NETFD the canonical quantization has not been fully established for relativistic fields yet although some attempts have been done.[14)] In the present paper it is assumed that the quantum correction is calculated by perturbative theory in the Fock space spanned by the ξ-oscillators. The thermal vacuum, $|\theta\rangle$, is defined by the transformed operator, ξ_p,

$$\langle\theta|\xi_p^\dagger = \xi_p|\theta\rangle = 0, \qquad \langle\theta|\tilde{\xi}_p^\dagger = \tilde{\xi}_p|\theta\rangle = 0. \tag{2·12}$$

The state, $|\theta\rangle$, is invariant under the tilde conjugation (2·4), (2·5) and (2·6),

$$(\langle\theta|)^\sim = \langle\theta|, \qquad (|\theta\rangle)^\sim = |\theta\rangle. \tag{2·13}$$

In equilibrium system the statistical thermal average is obtained by evaluating the expectation value under the thermal vacuum.

A neutral scalar field is expanded by using the a-oscillators, i.e. inverse of the thermal Bogoliubov transformations (2·8) on the ξ-oscillators,

$$\phi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p(t_x) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + a_p^\dagger(t_x) \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}), \tag{2·14}$$

$$\tilde{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (\tilde{a}_p(t_x) \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \tilde{a}_p^\dagger(t_x) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}). \tag{2·15}$$

From Eq. (2·7) the time dependence for the operators a and $a^\dagger$ is identical to $\tilde{a}^\dagger$ and $\tilde{a}$, respectively. Thus the thermal doublets notation is defined for the neutral scalar fields by

$$\begin{aligned}\phi^\alpha(x) &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left\{\begin{pmatrix} a_p(t_x) \\ \tilde{a}_p^\dagger(t_x)\end{pmatrix}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \begin{pmatrix} a_p^\dagger(t_x) \\ \tilde{a}_p(t_x)\end{pmatrix}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\right\} \\ &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left\{a_p^\alpha(t_x)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + (\tau_3\bar{a}_p(t_x)^T)^\alpha \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\right\}, \end{aligned} \tag{2·16}$$

$$\begin{aligned}\bar{\phi}^\alpha(x) &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left\{\begin{pmatrix} a_p^\dagger(t_x) & -\tilde{a}_p(t_x)\end{pmatrix}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \begin{pmatrix} a_p(t_x) & -\tilde{a}_p^\dagger(t_x)\end{pmatrix}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\right\} \\ &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left\{\bar{a}_p^\alpha(t_x)\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + (a_p(t_x)\tau_3)^\alpha \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\right\}, \end{aligned} \tag{2·17}$$

where τ_3 is the third Pauli matrix acting on the thermal doublets.

Below we express fields and propagators by the t-representation, which is defined by taking the spatial Fourier transform. A thermal propagator is defined by an expectation value of the time-ordered product of two fields in the thermal vacuum. It has a 2×2 matrix form with respect to the thermal index. For a neutral scalar field it is defined by

$$D^{\alpha\beta}(t_x - t_y; \boldsymbol{p}) \equiv \langle\theta|T[\phi^\alpha(t_x;\boldsymbol{p})\bar{\phi}^\beta(t_y;\boldsymbol{p})]|\theta\rangle, \tag{2·18}$$

where α and β are the thermal indices. Thus the thermal propagator is obtained by the thermal Bogoliubov transformation of the non-thermal one,[1),7),18),19)]

$$\begin{aligned}D^{\alpha\beta}(t_x - t_y;\boldsymbol{p}) &= B^{-1}(n_p(t_x))^{\alpha\gamma_1} D_{0,R}^{\gamma_1\gamma_2}(t_x-t_y;\boldsymbol{p})B(n_p(t_y))^{\gamma_2\beta} \\ &+\{\tau_3 B(n_p(t_x))^T\}^{\alpha\gamma_1} D_{0,A}^{\gamma_1\gamma_2}(t_x-t_y;\boldsymbol{p})\{B^{-1}(n_p(t_y))^T\tau_3\}^{\gamma_2\beta}, \end{aligned} \tag{2·19}$$

where $D_{0,R}^{\gamma_1\gamma_2}(t_x-t_y;\boldsymbol{p})$ and $D_{0,A}^{\gamma_1\gamma_2}(t_x-t_y;\boldsymbol{p})$ are the retarded and the advanced parts for the non-thermal propagator which are given by 2×2 matrices with thermal indices γ_1 and γ_2. The non-thermal propagators for the scalar field have diagonal forms. Each component in $D_{0,R}$ and $D_{0,A}$ is given by

$$D_{0,R}^{11}(t_x-t_y;\boldsymbol{p}) = \theta(t_x-t_y)\frac{1}{2\omega_p}\mathrm{e}^{-i\omega_p\cdot(t_x-t_y)}, \tag{2·20}$$

$$D_{0,R}^{22}(t_x-t_y;\boldsymbol{p}) = -\theta(t_y-t_x)\frac{1}{2\omega_p}\mathrm{e}^{-i\omega_p\cdot(t_x-t_y)}, \tag{2·21}$$

$$D_{0,A}^{11}(t_x-t_y;\boldsymbol{p}) = \theta(t_y-t_x)\frac{1}{2\omega_p}\mathrm{e}^{i\omega_p\cdot(t_x-t_y)}, \tag{2·22}$$

$$D_{0,A}^{22}(t_x-t_y;\boldsymbol{p}) = -\theta(t_x-t_y)\frac{1}{2\omega_p}\mathrm{e}^{i\omega_p\cdot(t_x-t_y)}, \tag{2·23}$$

other components $= 0$.

In NETFD we can choose the boundary condition for the thermal Bogoliubov matrices (2·11) to derive the ordinary Feynman rules for relativistic scalar fields.[1),17)]

It is convenient for perturbative calculations. In TFD the thermal average of the particle number density is given by the expectation value of the number operator $a^\dagger_{H,p}a_{H,p}$ in the thermal vacuum,

$$n_p(t) = \langle\theta_H|a^\dagger_{H,p}(t)a_{H,p}(t)|\theta_H\rangle, \tag{2·24}$$

where the fields and ground state with the lower index, H, are written in the Heisenberg representation. It coincides with the Bogoliubov parameter, n_p, in the Bogoliubov matrices (2·11) in an equilibrium state at tree level. Since quantum corrections can induce instability of the thermal vacuum (2·12), the Bogoliubov parameter does not always correspond to the particle number density out of equilibrium states. Below we assume that a stable vacuum is found by redefining the Bogoliubov parameter and the observed particle number density is obtained by the Bogoliubov parameter in the stable vacuum at the equal time limit.[1)]

§3. Time evolution equation

H. Umezawa and Y. Yamanaka have introduced the self-consistent renormalization condition to derive the Boltzmann equation in NETFD.[1),6)–8)] The hat-Hamiltonian (2·7) does not contain terms proportional to $\xi\tilde{\xi}$. However, such a term is induced through the quantum correction in the thermal self-energy out of equilibrium states. It breaks the condition (2·12) for the thermal vacuum. The terms proportional to $\xi\tilde{\xi}$ can be eliminated by the thermal Bogoliubov transformation. We impose that the terms proportional to $\xi\tilde{\xi}$ vanish at the equal time limit. The perturbed thermal Bogoliubov parameter can be fixed by this self-consistent renormalization condition. The Boltzmann equation appears as a consequence of this renormalization condition. It is also derived from diagonalizing the full propagator given by the Dyson equation at the equal time limit in non-relativistic quantum field theories.[11)] In this work we improve this diagonalization condition to be suitable for a relativistic scalar field.

In TFD the SD equation for the scalar field is given by

$$\begin{aligned}D_H^{\alpha\beta}(t_x-t_y;\boldsymbol{p}) &= D_0^{\alpha\beta}(t_x-t_y;\boldsymbol{p})\\ &+\int dt_{z_1}dt_{z_2}\ D_0^{\alpha\gamma_1}(t_x-t_{z_1};\boldsymbol{p})i\Sigma^{\gamma_1\gamma_2}(t_{z_1}-t_{z_2};\boldsymbol{p})D_H^{\gamma_2\beta}(t_{z_2}-t_y;\boldsymbol{p}),\end{aligned} \tag{3·1}$$

where $D_H^{\alpha\beta}(t_x-t_y;\boldsymbol{p})$ denotes the full thermal propagator, $D_0^{\alpha\beta}(t_x-t_y;\boldsymbol{p})$ the thermal propagator at tree level and $\Sigma^{\alpha\beta}(t_{z_1}-t_{z_2};\boldsymbol{p})$ the full thermal self-energy. Another expression for this equation is

$$\begin{aligned}D_H^{\alpha\beta}(t_x-t_y;\boldsymbol{p}) &= D_0^{\alpha\beta}(t_x-t_y;\boldsymbol{p})\\ &+\int dt_{z_1}dt_{z_2}\ D_H^{\alpha\gamma_1}(t_x-t_{z_1};\boldsymbol{p})i\Sigma^{\gamma_1\gamma_2}(t_{z_1}-t_{z_2};\boldsymbol{p})D_0^{\gamma_2\beta}(t_{z_2}-t_y;\boldsymbol{p}).\end{aligned} \tag{3·2}$$

We suppose that the full propagator can be decomposed in a similar form with the tree level one (2·19),

$$D_H^{\alpha\beta}(t_x-t_y;\boldsymbol{p}) = B^{-1}(n_{H,p}(t_x))^{\alpha\gamma_1}D_{H,R}^{\gamma_1\gamma_2}(t_x-t_y;\boldsymbol{p})B(n_{H,p}(t_y))^{\gamma_2\beta}$$

$$+\{\tau_3 B(n_{H,p}(t_x))^T\}^{\alpha\gamma_1} D_{H,A}^{\gamma_1\gamma_2}(t_x - t_y;\boldsymbol{p})\{B^{-1}(n_{H,p}(t_y))^T\tau_3\}^{\gamma_2\beta}, \tag{3·3}$$

where $n_{H,p}(t_x)$ and $n_{H,p}(t_y)$ are the Bogoliubov parameters acting on the full thermal propagator from the left- and the right-hand sides, respectively. For simplicity we omit the thermal indices and the momentum label $\boldsymbol{p}$ in the thermal propagator and the Bogoliubov parameters below.

We decompose the propagators in the SD equation (3·1) in accordance with Eqs. (2·19) and (3·3) to see the matrix structure of the thermal propagator. After the Klein-Gordon operator, $(\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2)$, is applied on the left to both sides, Eq. (3·1) reads

$$\begin{aligned}
&(\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2)\Big[B^{-1}(n_H(t_x))D_{H,R}(t_x - t_y)B(n_H(t_y)) \\
&\qquad +\tau_3 B(n_H(t_x))^T D_{H,A}(t_x - t_y)B^{-1}(n_H(t_y))^T\tau_3\Big] \\
&= (\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2)\Big[B^{-1}(n(t_x))D_{0,R}(t_x - t_y)B(n(t_y)) \\
&\qquad +\tau_3 B(n(t_x))^T D_{0,A}(t_x - t_y)B^{-1}(n(t_y))^T\tau_3\Big] \\
&\quad + \int dt_{z_1}dt_{z_2}\ (\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2)\Big[B^{-1}(n(t_x))D_{0,R}(t_x - t_{z_1})B(n(t_{z_1})) \\
&\qquad +\tau_3 B(n(t_x))^T D_{0,A}(t_x - t_{z_1})B^{-1}(n(t_{z_1}))^T\tau_3\Big] i\Sigma(t_{z_1} - t_{z_2}) \\
&\quad \times\Big[B^{-1}(n_H(t_{z_2}))D_{H,R}(t_{z_2} - t_y)B(n_H(t_y)) \\
&\qquad\quad +\tau_3 B(n_H(t_{z_2}))^T D_{H,A}(t_{z_2} - t_y)B^{-1}(n_H(t_y))^T\tau_3\Big],
\end{aligned} \tag{3·4}$$

where $n(t)$ is the Bogoliubov parameter for the non-perturbed operator included in the thermal propagator at tree level.

The Bogoliubov matrices and the Klein-Gordon operator do not commute. Substituting the expression (2·11) to the Bogoliubov matrices and inserting an identity operators, $B^{-1}(n_H)B(n_H)$, we obtain

$$\begin{aligned}
&B^{-1}(n_H(t_x))(\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2)\begin{pmatrix} D_{H,R}^{11}(t_x - t_y) & O_{R,prop1}(t_x, t_y) \\ 0 & D_{H,R}^{22}(t_x - t_y)\end{pmatrix} B(n_H(t_y)) \\
&+\tau_3 B(n_H(t_x))^T(\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2)\begin{pmatrix} D_{H,A}^{11}(t_x - t_y) & 0 \\ O_{A,prop1}(t_x, t_y) & D_{H,A}^{22}(t_x - t_y)\end{pmatrix} B(n_H(t_y))^T\tau_3 \\
&-\int dt_s\Bigg[B^{-1}(n_H(t_x))\begin{pmatrix} \frac{1}{2}\Sigma_R(t_x - t_s)D_{H,R}^{11}(t_s - t_y) & g_{x1}(t_x, t_y, t_s) \\ 0 & \frac{1}{2}\Sigma_A(t_x - t_s)D_{H,R}^{22}(t_s - t_y)\end{pmatrix} B(n_H(t_y)) \\
&+B^{-1}(n_H(t_x))\begin{pmatrix} g_{x2}(t_x, t_y, t_s) & -\frac{1}{2}\Sigma_R(t_x - t_s)D_{H,A}^{22}(t_s - t_y) \\ -\frac{1}{2}\Sigma_A(t_x - t_s)D_{H,A}^{11}(t_s - t_y) & 0\end{pmatrix} B^{-1}(n_H(t_y))^T\tau_3 \\
&+\tau_3 B(n_H(t_x))^T\begin{pmatrix} 0 & -\frac{1}{2}\Sigma_A(t_x - t_s)D_{H,R}^{22}(t_s - t_y) \\ -\frac{1}{2}\Sigma_R(t_x - t_s)D_{H,R}^{11}(t_s - t_y) & g_{x3}(t_x, t_y, t_s)\end{pmatrix} B(n_H(t_y)) \\
&+\tau_3 B(n_H(t_x))^T\begin{pmatrix} \frac{1}{2}\Sigma_A(t_x - t_s)D_{H,A}^{11}(t_s - t_y) & 0 \\ g_{x4}(t_x, t_y, t_s) & \frac{1}{2}\Sigma_R(t_x - t_s)D_{H,A}^{22}(t_s - t_y)\end{pmatrix} B(n_H(t_y))^T\tau_3\Bigg]
\end{aligned}$$

$$= -i\delta(t_x - t_y), \tag{3·5}$$

where the thermal self-energies for the retarded and the advanced propagators, Σ_R and Σ_A, are defined by

$$\Sigma_R \equiv \Sigma^{11} + \Sigma^{12} = \Sigma^{21} + \Sigma^{22}, \quad \Sigma_A \equiv \Sigma^{11} - \Sigma^{21} = \Sigma^{22} - \Sigma^{12}. \tag{3·6}$$

The off-diagonal elements, $O_{R(A),prop1}$, in the propagator (3·5) are defined to satisfy the following equations,

$$\begin{aligned}
&(\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2) O_{R,prop1}(t_x, t_y) \\
&\quad \equiv \ddot{n}_H(t_x) D_{H,R}^{22}(t_x - t_y) + 2\dot{n}_H(t_x)(\partial_{t_x} D_{H,R}^{22}(t_x - t_y)) \\
&\quad\quad - \ddot{n}(t_x) D_{0,R}^{22}(t_x - t_y) - 2\dot{n}(t_x)(\partial_{t_x} D_{0,R}^{22}(t_x - t_y)), \qquad (3·7) \\
&(\partial_{t_x}^2 + \boldsymbol{p}^2 + m^2) O_{A,prop1}(t_x, t_y) \\
&\quad \equiv -\ddot{n}_H(t_x) D_{H,A}^{11}(t_x - t_y) - 2\dot{n}_H(t_x)(\partial_{t_x} D_{H,A}^{11}(t_x - t_y)) \\
&\quad\quad + \ddot{n}(t_x) D_{0,A}^{11}(t_x - t_y) + 2\dot{n}(t_x)(\partial_{t_x} D_{0,A}^{11}(t_x - t_y)), \qquad (3·8)
\end{aligned}$$

We define the off-diagonal elements, $g_{x1} \sim g_{x4}$, by

$$\begin{aligned}
&g_{x1}(t_x, t_y, t_s) \\
&\equiv \frac{1}{2}\Big\{\Sigma^{12}(t_x - t_s) + h_-(t_x, t_s)\Big\} D_{H,R}^{22}(t_s - t_y) \\
&+ \int_{-\infty}^{\infty} dt_z \Bigg[\Big\{\ddot{n}(t_x) D_{0,R}^{22}(t_x - t_s) + 2\dot{n}(t_x)(\partial_{t_x} D_{0,R}^{22}(t_x - t_s))\Big\} \\
&\quad \times i\Sigma_A(t_s - t_z) D_{H,R}^{22}(t_s - t_y)\Bigg], \qquad (3·9) \\
&g_{x2}(t_x, t_y, t_s) \\
&\equiv \frac{1}{2}\Big\{\Sigma^{11}(t_x - t_s) + h_+(t_x, t_s)\Big\} D_{H,A}^{11}(t_s - t_y) \\
&- \int_{-\infty}^{\infty} dt_z \Bigg[\Big\{\ddot{n}(t_x) D_{0,R}^{22}(t_x - t_s) + 2\dot{n}(t_x)(\partial_{t_x} D_{0,R}^{22}(t_x - t_s))\Big\} \\
&\quad \times i\Sigma_A(t_s - t_z) D_{H,A}^{11}(t_z - t_y)\Bigg], \qquad (3·10) \\
&g_{x3}(t_x, t_y, t_s) \\
&\equiv -\frac{1}{2}\Big\{\Sigma^{22}(t_x - t_s) + h_+(t_x, t_s)\Big\} D_{H,R}^{22}(t_s - t_y) \\
&+ \int_{-\infty}^{\infty} dt_z \Bigg[\Big\{\ddot{n}(t_x) D_{0,A}^{11}(t_x - t_s) + 2\dot{n}(t_x)(\partial_{t_x} D_{0,A}^{11}(t_x - t_s))\Big\} \\
&\quad \times i\Sigma_A(t_s - t_z) D_{H,R}^{22}(t_s - t_y)\Bigg], \qquad (3·11)
\end{aligned}$$

$$
\begin{aligned}
&g_{x4}(t_x,t_y,t_s)\\
&\equiv -\frac{1}{2}\Big\{\Sigma^{21}(t_x-t_s)+h_-(t_x,t_s)\Big\}D^{11}_{H,A}(t_s-t_y)\\
&-\int_{-\infty}^{\infty}dt_z\Bigg[\Big\{\ddot{n}(t_x)D^{11}_{0,A}(t_x-t_s)+2\dot{n}(t_x)(\partial_{t_x}D^{11}_{0,A}(t_x-t_s))\Big\}\\
&\quad\times i\Sigma_A(t_s-t_z)D^{11}_{H,A}(t_s-t_y)\Bigg],
\end{aligned}
\tag{3·12}
$$

with

$$
h_-(t,t')\equiv n_H(t')\Sigma_R(t-t')-n_H(t)\Sigma_A(t-t'), \tag{3·13}
$$

$$
h_+(t,t')\equiv n_H(t')\Sigma_R(t-t')+n_H(t)\Sigma_A(t-t'). \tag{3·14}
$$

It should be noticed that the first and the second derivatives of the Bogoliubov parameters appear in Eq. (3·5) through the off-diagonal elements, $O_{R(A),prop1}$, and $g_{x1}\sim g_{x4}$.

We also rewrite the SD equation (3·2). Applying the Klein-Gordon operator from right to both sides of Eq. (3·2), we obtain

$$
\begin{aligned}
&\Big[B^{-1}(n_H(t_x))D_{H,R}(t_x-t_y)B(n_R(t_y))\\
&\quad+\tau_3B(n_H(t_x))^TD_{H,A}(t_x-t_y)B^{-1}(n_H(t_y))^T\tau_3\Big](\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)\\
&=\Big[B^{-1}(n(t_x))D_{0,R}(t_x-t_y)B(n(t_y))\\
&\quad+\tau_3B(n(t_x))^TD_{0,A}(t_x-t_y)B^{-1}(n(t_y))^T\tau_3\Big](\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)\\
&\quad+\int dt_{z_1}dt_{z_2}\Big[B^{-1}(n_H(t_x))D_{H,R}(t_x-t_{z_1})B(n_H(t_{z_1}))\\
&\qquad+\tau_3B(n_H(t_x))^TD_{H,A}(t_x-t_{z_1})B^{-1}(n_H(t_{z_1}))^T\tau_3\Big]\\
&\quad\times i\Sigma(t_{z_1}-t_{z_2})\Big[B^{-1}(n(t_{z_2}))D_{0,R}(t_{z_2}-t_y)B(n(t_y))\\
&\qquad+\tau_3B(n(t_{z_2}))^TD_{0,A}(t_{z_2}-t_y)B^{-1}(n(t_y))^T\tau_3\Big](\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2).
\end{aligned}
\tag{3·15}
$$

Substituting Eq. (2·11) to this equation, Eq. (3·15) reads

$$
\begin{aligned}
&B^{-1}(n_H(t_x))\begin{pmatrix} D^{11}_{H,R}(t_x-t_y) & O_{R,prop2}(t_x,t_y)\\ 0 & D^{22}_{H,R}(t_x-t_y)\end{pmatrix}(\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)B(n_H(t_y))\\
&+\tau_3B(n_H(t_x))^T\begin{pmatrix} D^{11}_{H,A}(t_x-t_y) & 0\\ O_{A,prop2}(t_x,t_y) & D^{22}_{H,A}(t_x-t_y)\end{pmatrix}(\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)B^{-1}(n_H(t_y))^T\tau_3\\
&-\int dt_s\Bigg[B^{-1}(n_H(t_x))\begin{pmatrix} \frac{1}{2}D^{11}_{H,R}(t_x-t_s)\Sigma_R(t_s-t_y) & g_{y1}(t_x,t_y,t_s)\\ 0 & \frac{1}{2}D^{22}_{H,R}(t_x-t_s)\Sigma_A(t_s-t_y)\end{pmatrix}B(n_H(t_y))\\
&+B^{-1}(n_H(t_x))\begin{pmatrix} g_{y2}(t_x,t_y,t_s) & -\frac{1}{2}D^{11}_{H,R}(t_x-t_s)\Sigma_R(t_s-t_y)\\ -\frac{1}{2}D^{22}_{H,R}(t_x-t_s)\Sigma_A(t_s-t_y) & 0\end{pmatrix}B^{-1}(n_H(t_y))^T\tau_3
\end{aligned}
$$

$$
\begin{aligned}
&+\tau_3 B(n_H(t_x))^T \begin{pmatrix} 0 & -\frac{1}{2}D^{11}_{H,A}(t_x-t_s)\Sigma_A(t_s-t_y) \\ -\frac{1}{2}D^{22}_{H,A}(t_x-t_s)\Sigma_R(t_s-t_y) & g_{y3}(t_x,t_y,t_s) \end{pmatrix} B(n_H(t_y)) \\
&+\tau_3 B(n_H(t_x))^T \begin{pmatrix} \frac{1}{2}D^{11}_{H,A}(t_x-t_s)\Sigma_A(t_s-t_y) & 0 \\ g_{y4}(t_x,t_y,t_s) & \frac{1}{2}D^{22}_{H,A}(t_x-t_s)\Sigma_R(t_s-t_y) \end{pmatrix} B^{-1}(n_H(t_y))^T\tau_3 \Bigg] \\
&= -i\delta(t_x-t_y),
\end{aligned} \tag{3·16}
$$

where the off-diagonal elements, $O_{R(A),prop2}$, are defined to satisfy

$$
\begin{aligned}
&O_{R,prop2}(t_x,t_y)(\overleftarrow{\partial}^{\,2}_{t_y}+\boldsymbol{p}^2+m^2) \\
&\quad = -\ddot{n}_H(t_y)D^{11}_{H,R}(t_x-t_y) - 2\dot{n}_H(t_y)(\partial_{t_y}D^{11}_{H,R}(t_x-t_y)) \\
&\qquad +\ddot{n}(t_y)D^{11}_{0,R}(t_x-t_y) + 2\dot{n}(t_y)(\partial_{t_y}D^{11}_{0,R}(t_x-t_y)),
\end{aligned} \tag{3·17}
$$

$$
\begin{aligned}
&O_{A,prop2}(t_x,t_y)(\overleftarrow{\partial}^{\,2}_{t_y}+\boldsymbol{p}^2+m^2) \\
&\quad = \ddot{n}_H(t_y)D^{22}_{H,A}(t_x-t_y) + 2\dot{n}_H(t_y)(\partial_{t_y}D^{22}_{H,A}(t_x-t_y)) \\
&\qquad -\ddot{n}(t_x)D^{22}_{0,A}(t_x-t_y) - 2\dot{n}(t_x)(\partial_{t_y}D^{22}_{0,A}(t_x-t_y)),
\end{aligned} \tag{3·18}
$$

and the off-diagonal elements, $g_{y1} \sim g_{y4}$, are

$$
\begin{aligned}
&g_{y1}(t_x,t_y,t_s) \\
&= \frac{1}{2}D^{11}_{H,R}(t_x-t_s)\Big\{\Sigma^{12}(t_s-t_y)+h_-(t_s,t_y)\Big\} \\
&- \int_{-\infty}^{\infty} dt_z \Bigg[D^{11}_{H,R}(t_x-t_z)i\Sigma_R(t_z-t_s) \\
&\quad \times\Big\{\ddot{n}(t_y)D^{11}_{0,R}(t_s-t_y)+2\dot{n}(t_y)(\partial_{t_y}D^{11}_{0,R}(t_s-t_y))\Big\}\Bigg],
\end{aligned} \tag{3·19}
$$

$$
\begin{aligned}
&g_{y2}(t_x,t_y,t_s) \\
&= \frac{1}{2}D^{11}_{H,R}(t_x-t_s)\Big\{\Sigma^{11}(t_s-t_y)+h_+(t_s,t_y)\Big\} \\
&- \int_{-\infty}^{\infty} dt_z \Bigg[D^{11}_{H,R}(t_x-t_z)i\Sigma_R(t_z-t_s) \\
&\quad \times\Big\{\ddot{n}(t_y)D^{22}_{0,A}(t_s-t_y)+2\dot{n}(t_y)(\partial_{t_y}D^{22}_{0,A}(t_s-t_y))\Big\}\Bigg],
\end{aligned} \tag{3·20}
$$

$$
\begin{aligned}
&g_{y3}(t_x,t_y,t_s) \\
&= -\frac{1}{2}D^{22}_{H,A}(t_x-t_s)\Big\{\Sigma^{22}(t_s-t_y)+h_+(t_s,t_y)\Big\} \\
&+ \int_{-\infty}^{\infty} dt_z \Bigg[D^{22}_{H,A}(t_x-t_z)i\Sigma_R(t_z-t_s) \\
&\quad \times\Big\{\ddot{n}(t_y)D^{11}_{0,R}(t_s-t_y)+2\dot{n}(t_y)(\partial_{t_y}D^{11}_{0,R}(t_s-t_y))\Big\}\Bigg],
\end{aligned} \tag{3·21}
$$

$$
\begin{aligned}
&g_{y4}(t_x,t_y,t_s)\\
&=-\frac{1}{2}D^{22}_{H,A}(t_x-t_s)\Big\{\Sigma^{21}(t_s-t_y)+h_-(t_s,t_y)\Big\}\\
&+\int_{-\infty}^{\infty}dt_z\Bigg[D^{22}_{H,A}(t_x-t_z)i\Sigma_R(t_z-t_s)\\
&\quad\times\Big\{\ddot{n}(t_y)D^{22}_{0,A}(t_s-t_y)+2\dot{n}(t_y)(\partial_{t_y}D^{22}_{0,A}(t_s-t_y))\Big\}\Bigg].
\end{aligned}
\tag{3·22}
$$

For non-relativistic fields the particle number conservation law is derived from the difference between the SD equation applying the differential operator, $i\partial_{t_x}+\boldsymbol{p}^2/2m$, on the left and right sides at the equal time limit.[21] We adapt the procedure to relativistic scalar fields. Subtracting the Eq. (3·16) from Eq. (3·5), we obtain

$$
\begin{aligned}
&B^{-1}(n_H(t_x))\Bigg[(\partial^2_{t_x}+\boldsymbol{p}^2+m^2)\begin{pmatrix} D^{11}_{H,R}(t_x-t_y) & O_{R,prop1}(t_x,t_y)\\ 0 & D^{22}_{H,R}(t_x-t_y)\end{pmatrix}\\
&\quad-\begin{pmatrix} D^{11}_{H,R}(t_x-t_y) & O_{R,prop2}(t_x,t_y)\\ 0 & D^{22}_{H,R}(t_x-t_y)\end{pmatrix}(\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)\\
&\quad-\int dt_s\Bigg\{\begin{pmatrix} \frac{1}{2}\Sigma_R(t_x-t_s)D^{11}_{H,R}(t_s-t_y) & g_{x1}(t_x,t_y,t_s)\\ 0 & \frac{1}{2}\Sigma_A(t_x-t_s)D^{22}_{H,R}(t_s-t_y)\end{pmatrix}\\
&\qquad-\begin{pmatrix} \frac{1}{2}D^{11}_{H,R}(t_x-t_s)\Sigma_R(t_s-t_y) & g_{y1}(t_x,t_y,t_s)\\ 0 & \frac{1}{2}D^{22}_{H,R}(t_x-t_s)\Sigma_A(t_s-t_y)\end{pmatrix}\Bigg\}\Bigg]B(n_H(t_y))\\
&+\tau_3B(n_H(t_x))^T\Bigg[(\partial^2_{t_x}+\boldsymbol{p}^2+m^2)\begin{pmatrix} D^{11}_{H,A}(t_x-t_y) & 0\\ O_{A,prop1}(t_x,t_y) & D^{22}_{H,A}(t_x-t_y)\end{pmatrix}\\
&\quad-\begin{pmatrix} D^{11}_{H,A}(t_x-t_y) & 0\\ O_{A,prop2}(t_x,t_y) & D^{22}_{H,A}(t_x-t_y)\end{pmatrix}(\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)\\
&\quad-\int dt_s\Bigg\{\begin{pmatrix} \frac{1}{2}\Sigma_A(t_x-t_s)D^{11}_{H,A}(t_s-t_y) & 0\\ g_{x4}(t_x,t_y,t_s) & \frac{1}{2}\Sigma_R(t_x-t_s)D^{22}_{H,A}(t_s-t_y)\end{pmatrix}\\
&\qquad-\begin{pmatrix} \frac{1}{2}D^{11}_{H,A}(t_x-t_s)\Sigma_A(t_s-t_y) & 0\\ g_{y4}(t_x,t_y,t_s) & \frac{1}{2}D^{22}_{H,A}(t_x-t_s)\Sigma_R(t_s-t_y)\end{pmatrix}\Bigg\}\Bigg]B^{-1}(n_H(t_y))^T\tau_3\\
&-B^{-1}(n_H(t_x))\int dt_s\Bigg[\begin{pmatrix} g_{x2}(t_x,t_y,t_s) & -\frac{1}{2}\Sigma_R(t_x-t_s)D^{22}_{H,A}(t_s-t_y)\\ -\frac{1}{2}\Sigma_A(t_x-t_s)D^{11}_{H,A}(t_s-t_y) & 0\end{pmatrix}\\
&\quad-\begin{pmatrix} g_{y2}(t_x,t_y,t_s) & -\frac{1}{2}D^{11}_{H,R}(t_x-t_s)\Sigma_R(t_s-t_y)\\ -\frac{1}{2}D^{22}_{H,R}(t_x-t_s)\Sigma_A(t_s-t_y) & 0\end{pmatrix}\Bigg]B^{-1}(n_H(t_y))^T\tau_3\\
&-\tau_3B(n_H(t_x))^T\int dt_s\Bigg[\Bigg(\begin{matrix} 0 & -\frac{1}{2}\Sigma_A(t_x-t_s)D^{22}_{H,R}(t_s-t_y)\\ -\frac{1}{2}\Sigma_R(t_x-t_s)D^{11}_{H,R}(t_s-t_y) & g_{x3}(t_x,t_y,t_s)\end{matrix}\Bigg)
\end{aligned}
$$

$$-\left(\begin{array}{cc} 0 & -\frac{1}{2}D^{11}_{H,A}(t_x-t_s)\Sigma_A(t_s-t_y) \\ -\frac{1}{2}D^{22}_{H,A}(t_x-t_s)\Sigma_R(t_s-t_y) & g_{y3}(t_x,t_y,t_s) \end{array}\right)\Bigg]B(n_H(t_y))=0. \quad (3\cdot23)$$

Each elements of the matrices between the Bogoliubov matrices $\tau_3 B^T(\cdots)B$ and $B^{-1}(\cdots)B^{-1T}\tau_3$ and the diagonal element in $B^{-1}(\cdots)B$ and $\tau_3 B^T(\cdots)B^{-1T}\tau_3$ satisfies trivial equation, $0=0$, at the equal time limit, $t_x \to t_y$, in the case of the relativistic scalar field with a four-point self-interaction, as is shown in the next section. The remaining off-diagonal elements should satisfy

$$\lim_{t_x\to t_y}\Big[(\partial^2_{t_x}+\boldsymbol{p}^2+m^2)O_{R,prop1}(t_x,t_y)-O_{R,prop2}(t_x,t_y)(\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)$$
$$-\int dt_s\ \Big\{g_{x1}(t_x,t_y,t_s)-g_{y1}(t_x,t_y,t_s)\Big\}\Big]=0, \quad (3\cdot24)$$
$$\lim_{t_x\to t_y}\Big[(\partial^2_{t_x}+\boldsymbol{p}^2+m^2)O_{A,prop1}(t_x,t_y)-O_{A,prop2}(t_x,t_y)(\overleftarrow{\partial}^2_{t_y}+\boldsymbol{p}^2+m^2)$$
$$-\int dt_s\ \Big\{g_{x4}(t_x,t_y,t_s)-g_{y4}(t_x,t_y,t_s)\Big\}\Big]=0. \quad (3\cdot25)$$

Substituting Eqs. (3·7), (3·9), (3·17) and (3·19) into Eq. (3·24), we obtain

$$\begin{aligned}&\lim_{t_x\to t_y}\Big(\ddot{n}_H(t_x)D^{22}_{H,R}(t_x-t_y)-\ddot{n}(t_x)D^{22}_{0,R}(t_x-t_y)\\ &+\ddot{n}_H(t_y)D^{11}_{H,R}(t_x-t_y)-\ddot{n}(t_y)D^{11}_{0,R}(t_x-t_y)\\ &+2\dot{n}_H(t_x)(\partial_{t_x}D^{22}_{H,R}(t_x-t_y))-2\dot{n}(t_x)(\partial_{t_x}D^{22}_{0,R}(t_x-t_y))\\ &+2\dot{n}_H(t_y)(\partial_{t_y}D^{11}_{H,R}(t_x-t_y))-2\dot{n}(t_y)(\partial_{t_y}D^{11}_{0,R}(t_x-t_y))\\ &+\frac{i}{2}\int_{-\infty}^{\infty}dt_s\Big[\Big\{i\Sigma^{12}(t_x-t_s)+ih_-(t_x,t_s)\Big\}D^{22}_{H,R}(t_s-t_y)\\ &-D^{11}_{H,R}(t_x-t_s)\Big\{i\Sigma^{12}(t_s-t_y)+ih_-(t_s,t_y)\Big\}\Big]\Big)=0. \end{aligned} \quad (3\cdot26)$$

This equation describes the time evolution of the Bogoliubov parameter, $n(t)$.

We write the differences between the perturbed and the unperturbed Bogoliubov parameter as

$$\nu(t)\equiv n_H(t)-n(t). \quad (3\cdot27)$$

Substituting Eqs. (A·13)-(A·16) to Eq. (3·26), a delta function , $\delta(t_x-t_y)$, appears from the first derivative of the propagator. It diverges at the equal time limit. The divergence is canceled out in Eq. (3·26). The ordinary Boltzmann equation does not contain terms proportional to the second derivative of the thermal Bogoliubov parameter, $\ddot{\nu}$. These terms in Eq. (3·26) are canceled out at the equal time limit. Therefore the equation (3·26) is simplified to a Markovian equation,

$$\begin{aligned}\dot{\nu}(t_x)=-\frac{1}{2}\lim_{t_x\to t_y}\int dt_s\Big[&\Big\{i\Sigma^{12}(t_x-t_s)+ih_-(t_x,t_s)\Big\}D^{22}_{H,R}(t_s-t_y)\\ &-D^{11}_{H,R}(t_x-t_s)\Big\{i\Sigma^{12}(t_s-t_y)+ih_-(t_s,t_y)\Big\}\Big].\end{aligned} \quad (3\cdot28)$$

In a similar manner Eq. (3·25) reduces to a Markovian equation for the advanced propagator,

$$\dot{\nu}(t_x) = -\frac{1}{2}\lim_{t_x \to t_y}\int dt_s \Big[\Big\{i\Sigma^{21}(t_x - t_s) + ih_-(t_x, t_s)\Big\}D^{11}_{H,A}(t_s - t_y) \\ -D^{22}_{H,A}(t_x - t_s)\Big\{i\Sigma^{21}(t_s - t_y) + ih_-(t_s, t_y)\Big\}\Big]. \qquad (3·29)$$

Therefore the time evolution of the Bogoliubov parameter is determined by solving Eqs. (3·28) and (3·29).

§4. Boltzmann equation in a $\lambda\phi^4$ theory

It is expected that the time evolution of the number distribution is given by the Boltzmann equation in NETFD. In this section we show that the Boltzmann equation is obtained from Eqs. (3·28) and (3·29). We consider a neutral scalar field with a four-point self-interaction in (1+2) dimensions, for simplicity. To derive the Boltzmann equation we calculate the self-energy and evaluate the time evolution equations (3·28) and (3·29). We start from the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{1}{2}m^2\phi(x)^2 - \frac{\lambda}{4!}\phi(x)^4. \qquad (4·1)$$

In TFD the total Lagrangian density, $\hat{\mathcal{L}}$, for non-tilde and tilde fields is given by

$$\hat{\mathcal{L}}(x) \equiv \mathcal{L}(x) - \tilde{\mathcal{L}}(x), \qquad (4·2)$$

where $\tilde{\mathcal{L}}$ is the tilde conjugate of the Lagrangian density $\mathcal{L}$. In this model the self-energy at 1-loop level can not contribute to the right-hand sides in Eqs. (3·28) and (3·29). We have to calculate the self-energy at the 2-loop level.

The self-energy is calculated by using the Feynman rules in TFD.[14),20)] In the thermal doublet notation the Feynman rule for the four-point scalar vertex is given by a vector like coupling constant which consists of the coupling constants for the non-tilde and the tilde fields. Thus the following factor is assigned to the ϕ^4 vertex,

$$\lambda^\alpha = \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \qquad (4·3)$$

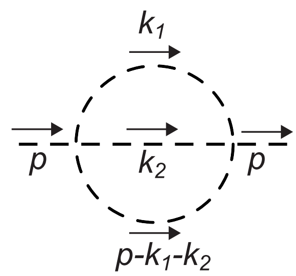

Fig. 1. 2-loop thermal self-energy in $\lambda\phi^4$ interacting model

In Fig. 1 we show the lowest order Feynman diagram which contributes to the time evolution equation for the neutral scalar field. According to the thermal propagator (2·19) and the coupling constant (4·3), the self-energy is given by

$$\begin{aligned}
&i\Sigma_B^{\gamma_1\gamma_2}(t_{z_1}-t_{z_2};\boldsymbol{p})\\
&=-\frac{\lambda^2}{3!}\int\frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}D_0^{\gamma_1\gamma_2}(t_{z_1}-t_{z_2},\boldsymbol{k}_1)D_0^{\gamma_1\gamma_2}(t_{z_1}-t_{z_2},\boldsymbol{k}_2)D_0^{\gamma_1\gamma_2}(t_{z_1}-t_{z_2},\boldsymbol{q}),
\end{aligned} \tag{4·4}$$

with $\boldsymbol{q}\equiv\boldsymbol{p}-\boldsymbol{k}_1-\boldsymbol{k}_2$. Substituting the explicit expression for the thermal propagator (2·19), we obtain

$$\begin{aligned}
&i\Sigma_B^{\gamma_1\gamma_2}(t_{z_1}-t_{z_2};\boldsymbol{p})\\
&=-\frac{\lambda^2}{3!}\sum_{i1=1}^{2}\sum_{i2=1}^{2}\sum_{i3=1}^{2}\int\frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}\frac{1}{8\omega_{k_1}\omega_{k_2}\omega_q} \qquad (4·5)\\
&\times\Bigg[\theta(t_{z_1}-t_{z_2})e^{i(E_{k_1,i1}+E_{k_2,i2}+E_{q,i3})(t_{z_1}-t_{z_2})}\\
&\times\begin{pmatrix} f_{k_1,i_1,a}(t_{z_2})f_{k_2,i_2,a}(t_{z_2})f_{q,i_3,a}(t_{z_2}) & -f_{k_1,i_1,b}(t_{z_2})f_{k_2,i_2,b}(t_{z_2})f_{q,i_3,b}(t_{z_2})\\ f_{k_1,i_1,a}(t_{z_2})f_{k_2,i_2,a}(t_{z_2})f_{q,i_3,a}(t_{z_2}) & -f_{k_1,i_1,b}(t_{z_2})f_{k_2,i_2,b}(t_{z_2})f_{q,i_3,b}(t_{z_2})\end{pmatrix}\\
&+\theta(t_{z_2}-t_{z_1})e^{-i(E_{k_1,i_1}+E_{k_2,i_2}+E_{q,i_3})(t_{z_1}-t_{z_2})}\\
&\times\begin{pmatrix} f_{k_1,i_1,a}(t_{z_1})f_{k_2,i_2,a}(t_{z_1})f_{q,i_3,a}(t_{z_1}) & -f_{k_1,i_1,a}(t_{z_1})f_{k_2,i_2,a}(t_{z_1})f_{q,i_3,a}(t_{z_1})\\ f_{k_1,i_1,b}(t_{z_1})f_{k_2,i_2,b}(t_{z_1})f_{q,i_3,b}(t_{z_1}) & -f_{k_1,i_1,b}(t_{z_1})f_{k_2,i_2,b}(t_{z_1})f_{q,i_3,b}(t_{z_1})\end{pmatrix}\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
&E_{q,1}\equiv\omega_q,\quad E_{q,2}\equiv-\omega_q,\\
&f_{q,1,a}(t)\equiv n_q(t),\quad f_{q,2,a}(t)\equiv 1+n_q(t),\\
&f_{q,1,b}(t)\equiv 1+n_q(t),\quad f_{q,2,b}(t)\equiv n_q(t).
\end{aligned} \tag{4·6}$$

We notice that unperturbed Bogoliubov parameters appear in the internal lines.

The time dependent part in Eq. (4·5) has the following form,

$$V(t-t')=\theta(\pm(t-t'))\mathrm{e}^{\pm iW(t-t')}. \tag{4·7}$$

It is rewritten in a Fourier integral form,

$$V(t-t')=-i\int\frac{dp_0}{2\pi}\frac{1}{p_0\mp W\mp i\varepsilon}\mathrm{e}^{-ip_0(t-t')}. \tag{4·8}$$

Above p_0 integral is simplified in the on-shell approximation. We set p_0 on the denominator in Eq. (4·8) to the on-shell value, $p_0=\omega_p$.[1)] Thus we can perform the Fourier integration and obtain

$$V(t-t')=-i\delta(t-t')\frac{1}{\omega_p\mp W\mp i\varepsilon}. \tag{4·9}$$

It is decomposed into the real and the imaginary parts.

$$V(t-t') = -i\delta(t-t')\left(\mathrm{P}\frac{1}{\omega_p \mp W} \pm 2\pi i\delta^0(\omega_p - W)\right), \tag{4·10}$$

where P denotes the principal part of $1/(\omega_p \mp W)$. Under the on-shell approximation the 2-loop thermal self-energy (4·5) simplifies to

$$\begin{aligned}
& i\Sigma_B^{\gamma_1\gamma_2}(t-t';\boldsymbol{p}) \\
& = -i\frac{\lambda^2}{3!}\delta(t-t')\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\int\frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}\frac{1}{8\omega_{k_1}\omega_{k_2}\omega_q} \\
& \times\Bigg[\frac{1}{\omega_p + E_{k_1,i_1} + E_{k_2,i_2} + E_{q,i_3} + i\epsilon} \\
& \times\begin{pmatrix} f_{k_1,i_1,a}(t')f_{k_2,i_2,a}(t')f_{q,i_3,a}(t') & -f_{k_1,i_1,b}(t')f_{k_2,i_2,b}(t')f_{q,i_3,b}(t') \\ f_{k_1,i_1,a}(t')f_{k_2,i_2,a}(t')f_{q,i_3,a}(t') & -f_{k_1,i_1,b}(t')f_{k_2,i_2,b}(t')f_{q,i_3,b}(t') \end{pmatrix} \\
& -\frac{1}{\omega_p - E_{k_1,i_1} - E_{k_2,i_2} - E_{q,i_3} - i\epsilon} \\
& \times\begin{pmatrix} f_{k_1,i_1,a}(t)f_{k_2,i_2,a}(t)f_{q,i_3,a}(t) & -f_{k_1,i_1,a}(t)f_{k_2,i_2,a}(t)f_{q,i_3,a}(t) \\ f_{k_1,i_1,b}(t)f_{k_2,i_2,b}(t)f_{q,i_3,b}(t) & -f_{k_1,i_1,b}(t)f_{k_2,i_2,b}(t)f_{q,i_3,b}(t) \end{pmatrix}\Bigg].
\end{aligned} \tag{4·11}$$

The perturbed propagator, D_H, is necessary to evaluate the right-hand side in Eqs. (3·28) and (3·29). Here we drop the higher order corrections and use the unperturbed propagator, D_0, instead of the perturbed one, D_H. Substituting the thermal self-energy (4·5) into the time evolution equations for the Bogoliubov parameter, (3·28) and (3·29) and replacing the perturbed propagator with the unperturbed one, we derive a equation with the structure of the Boltzmann equation for the $\lambda\phi^4$ interaction model. We will call it Boltzmann equation,

$$\begin{aligned}
\dot{\nu}_p(t_x) &= \frac{\lambda^2}{3!}\sum_{i1=1}^{2}\sum_{i2=1}^{2}\sum_{i3=1}^{2}\int_{-\infty}^{t_x}dt_s\int\frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}\frac{1}{16\omega_p\omega_{k_1}\omega_{k_2}\omega_q} \\
&\times\cos\big\{(-\omega_p + E_{k_1,i_1} + E_{k_2,i_2} + E_{q,i_3})(t_x - t_s)\big\} \\
&\quad\times\Big[(1+n_{H,p}(t_s))f_{k_1,i_1,a}(t_s)f_{k_2,i_2,a}(t_s)f_{q,i_3,a}(t_s) \\
&\quad - n_{H,p}(t_s)f_{k_1,i_1,b}(t_s)f_{k_2,i_2,b}(t_s)f_{q,i_3,b}(t_s)\Big].
\end{aligned} \tag{4·12}$$

It should be noticed that both Eqs. (3·28) and (3·29) provide the same expression (4·12). In the on-shell approximation the principal part in Eq. (4·10) is canceled out from the time evolution equation. Thus the Boltzmann equation (4·12) reduces to

$$\dot{\nu}_p(t_x) = \frac{\lambda^2}{3!}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\int\frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}\frac{\pi}{8\omega_p\omega_{k_1}\omega_{k_2}\omega_q}$$

$$\times\delta(\omega_p - E_{k_1,i_1} - E_{k_2,i_2} - E_{q,i_3})$$
$$\times\Big[(1 + n_{H,p}(t_x))f_{k_1,i_1,a}(t_x)f_{k_2,i_2,a}(t_x)f_{q,i_3,a}(t_x)$$
$$- n_{H,p}(t_x)f_{k_1,i_1,b}(t_x)f_{k_2,i_2,b}(t_x)f_{q,i_3,b}(t_x)\Big]. \quad (4{\cdot}13)$$

The right-hand side of this equation contains the delta function for the energy conservation and the statistical factors. It corresponds to the collision term in the Boltzmann equation.

§5. Numerical analysis of the Boltzmann equation

The Boltzmann equation (4·12) describes the time evolution of the particle number distribution. We numerically solve it starting from the Bose distribution with the temperature, T, and the mass, m_0,

$$n_p(t = 0) = \frac{1}{e^{\sqrt{\boldsymbol{p}^2+m_0^2}/T} - 1}. \quad (5{\cdot}1)$$

It is assumed that the scalar mass suddenly changes from m_0 to m. Then the state is no longer in equilibrium. Below we set parameters to $m_0 = 5 \times 10^{-2}\mu$, $m = 4 \times 10^{-2}\mu$, $T = \mu$ and $\lambda = \mu$, with an arbitrary mass scale, μ. It is expected that the distribution function, $n_p(t)$, approaches to the Bose distribution, $n_{f,p}$, with temperature, T_f. The final state temperature can be estimated at $T_f = 0.988\mu$ from the energy conservation law.

First we numerically solve the Boltzmann equation (4·13) under the on-shell approximation. It is the differential equation in terms of the time variable. We employ the fourth order Runge-Kutta algorithm to solve it. In each steps of Runge-Kutta algorithm the momentum integral is performed by the second order Simpson integral. The radial part of the momentum integral is cut off at the scale, $\Lambda = 20\mu$. In order to reduce the numerical error the integral interval is divided into two sub intervals, $[0, 0.3\Lambda]$ and $[0.3\Lambda, \Lambda]$. The Simpson's rule is applied to each sub interval. The scattering with the particle in the final equilibrium state is introduced through the self-energy.

For the numerical analysis we define the deviation from the final equilibrium distribution with the temperature, T_f, by

$$\delta n_p(t) = n_p(t) - n_{f,p}. \quad (5{\cdot}2)$$

We expand the distribution function $f(t)$ in Eq. (4·13) in terms of δn,

$$f_{q,i,a(b)}(t) = f^f_{q,i,a(b)} + O(\delta n(t)), \quad (5{\cdot}3)$$

where $i = 1, 2$ and

$$f^f_{q,1,a} = f^f_{q,2,b} = n_{f,q}, \quad (5{\cdot}4)$$
$$f^f_{q,2,a} = f^f_{q,1,b} = 1 + n_{f,q}. \quad (5{\cdot}5)$$

In our setup the initial state temperature, $T = \mu$, is close to the expected final state temperature, $T_f = 0.988\mu$. Then we keep only the leading order δn expansion in numerical calculations. The Bogoliubov parameters is fixed to $n_{f,p}$ for the internal lines in the self-energy. We solve the time evolution of the one for the external line, n_H.

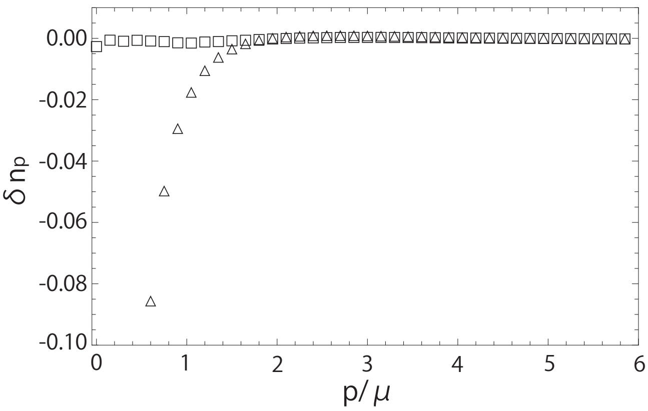

Fig. 2. Behavior of $\delta n_p(t)$ in the on-shell approximation (4·13). The triangle and square points show $\delta n_p(t = 0)$ and $\delta n_p(t = 1.0 \times 10^2 \mu^{-1})$, respectively.

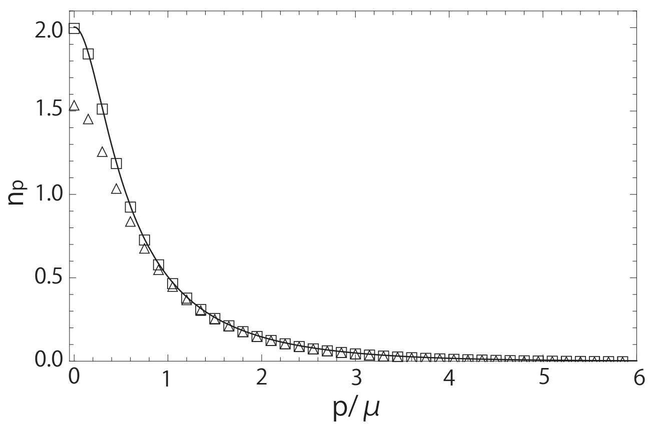

Fig. 3. Behavior of $n_p(t)$ in the on-shell approximation (4·13). The triangle and square points show $n_p(t = 0)$ and $n_p(t = 1.0 \times 10^2 \mu^{-1})$, respectively. The solid line represents the Bose distribution at $T = T_f$.

Evaluating the Boltzmann equation (4·13) in above assumptions, we obtain the time evolution of the distribution function. In Fig. 2 the behavior of $\delta n_p(t)$ is shown as a function of the momentum p. At $t = 0$ the particle number density with a lower momentum is much smaller than the estimated final one. We also draw the behavior of the particle number distribution, $n_p(t)$, in Fig. 3. As is expected, the particle number distribution approaches the equilibrium distribution with the temperature, T_f.

To evaluate the off-shell mode contribution we rewrite Eq. (4·12) as

$$\int_{-\infty}^{t_x} dt_s\ \dot{\nu}(t_s) = \frac{\lambda^2}{3!}\sum_{i1=1}^{2}\sum_{i2=1}^{2}\sum_{i3=1}^{2}\int_{-\infty}^{t_x} dt_s \int \frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}\frac{1}{16\omega_p\omega_{k_1}\omega_{k_2}\omega_q}$$
$$\times\frac{\sin\left\{(-\omega_p+E_{k_1,i1}+E_{k_2,i2}+E_{q,i3})(t_x-t_s)\right\}}{-\omega_p+E_{k_1,i1}+E_{k_2,i2}+E_{q,i3}}$$
$$\times\Big[(1+n_{H,p}(t_s))f_{k_1,i1,a}(t_s)f_{k_2,i2,a}(t_s)f_{q,i3,a}(t_s)$$
$$-\,n_{H,p}(t_s)f_{k_1,i1,b}(t_s)f_{k_2,i2,b}(t_s)f_{q,i3,b}(t_s)\Big]. \tag{5·6}$$

The Boltzmann equation (4·12) is reproduced by differentiating both the sides of this equation in terms of t_x. This equation is satisfied for an arbitrary t_x. Thus we obtain the differential equation,

$$\dot{\nu}(t_s) = \frac{\lambda^2}{3!}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\int \frac{d^2\boldsymbol{k}_1}{(2\pi)^2}\frac{d^2\boldsymbol{k}_2}{(2\pi)^2}\frac{1}{16\omega_p\omega_{k_1}\omega_{k_2}\omega_q}$$
$$\times\frac{\sin\left\{(-\omega_p+E_{k_1,i_1}+E_{k_2,i_2}+E_{q,i_3})(t_x-t_s)\right\}}{-\omega_p+E_{k_1,i_1}+E_{k_2,i_2}+E_{q,i_3}}$$
$$\times\Big[(1+n_{H,p}(t_s))f_{k_1,i_1,a}(t_s)f_{k_2,i_2,a}(t_s)f_{q,i_3,a}(t_s)$$
$$-\,n_{H,p}(t_s)f_{k_1,i1,b}(t_s)f_{k_2,i2,b}(t_s)f_{q,i3,b}(t_s)\Big]. \tag{5·7}$$

The time integral is dropped. Therefore the same numerical algorithm for the on-shell approximation can be applied to Eq. (5·7). The time variable, t_x, in Eq. (5·7) represents the time interval from the initial moment. Here we set $t_x = 1.0\times 10^3\mu^{-1}$. At the limit, $t_x\to\infty$, the off-shell mode contribution disappears.

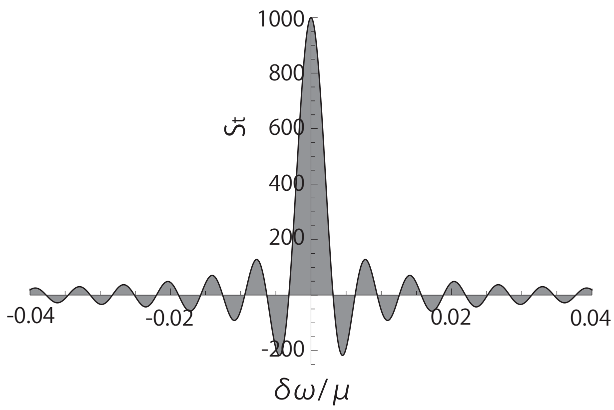

Fig. 4. Behavior of $S_t(\delta\omega)$ for $t_x - t_s = 1000\mu^{-1}$.

The integral kernel in Eq. (5·7) is proportional to

$$S_t(\delta\omega)\equiv\frac{\sin\{\delta\omega(t_x-t_s)\}}{\delta\omega}. \tag{5·8}$$

As is shown in Fig. 4, it has a peak at the on-shell limit, $\delta\omega \to 0$, and frequently oscillates for a larger $\delta\omega$. Thus it is enough to evaluate the integral near the on-shell limit. We restrict the integral interval, $-12\pi \leq \delta\omega(t_x - t_s) \leq 12\pi$.

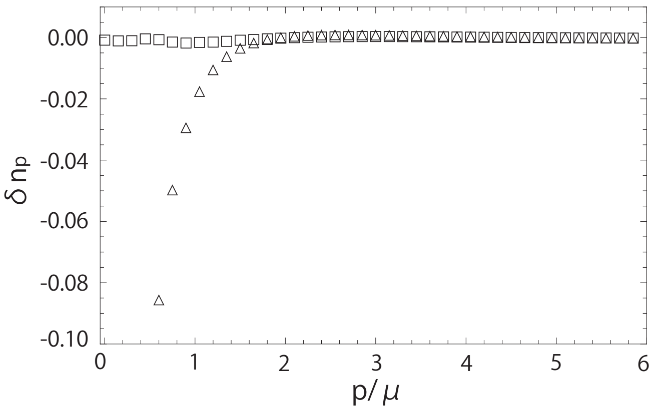

Fig. 5. Behavior of $\delta n_p(t)$ with the off-shell contribution (5·7). The triangle and square points show $\delta n_p(t=0)$ and $\delta n_p(t = 1.0 \times 10^2 \mu^{-1})$, respectively.

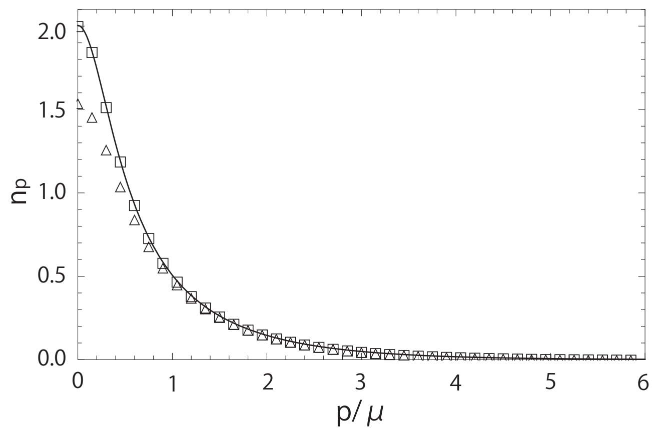

Fig. 6. Behavior of $n_p(t)$ with the off-shell contribution (5·7). The triangle and square points show $n_p(t=0)$ and $n_p(t = 1.0 \times 10^2 \mu^{-1})$, respectively. The solid line represents the Bose distribution at $T = T_f$.

We numerically evaluate the Boltzmann equation (5·7) in above approximation and obtain the time evolution of the distribution function with the off-shell mode contribution. The behavior of $\delta n_p(t)$ and $n_p(t)$ is illustrated as a function of the momentum p in Figs. 5 and 6, respectively. Hence, a similar behavior is observed for Eqs. (4·13) and (5·7). It seems to be difficult to distinguish the contribution from the off-shell mode in these figures.

To evaluate the contribution from the off-shell mode we calculate the relaxation time, τ_p. According to the linear response theory, we define it by

$$\tau_p \equiv -\frac{\delta n_p}{\delta \dot{n}_p}. \tag{5·9}$$

It can be calculated by solving the Boltzmann equations (4·13) and (5·7).

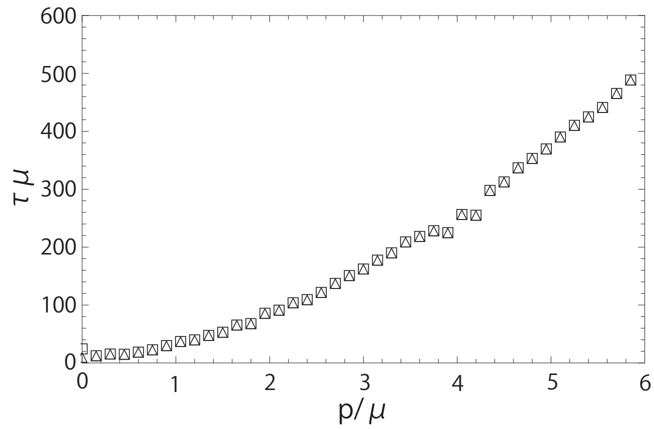

Fig. 7. The relaxation time, τ_p, in the on-shell approximation. The triangle and square points show $\tau(t = 0)$ and $\tau(t = 1.0 \times 10^2\mu^{-1})$, respectively.

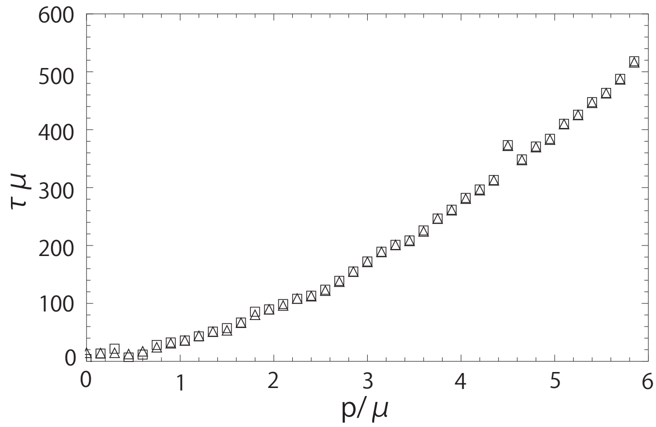

Fig. 8. The relaxation time, τ_p, with the off-shell contribution. The triangle and square points show $\tau(t = 0)$ and $\tau(t = 1.0 \times 10^2\mu^{-1})$, respectively.

We plot the behavior of the relaxation time, τ_p, as a function of the momentum, p, in Fig. 7 under the on-shell approximation. The off-shell contribution is included in Fig. 8. The relaxation time is almost constant with respect to the time variable, t. A longer relaxation time is observed for a higher momentum mode. As is shown in Figs. 3 and 6 the distribution function $n(t)$ is close to the final equilibrium state at $t = 1.0 \times 10^2\mu^{-1}$. It is consistent with the behavior of the relaxation time. For $p \lesssim 2.0\mu$ the relaxation time is smaller than $1.0 \times 10^2\mu^{-1}$. The particle number distribution for a higher momentum mode is close to the final equilibrium state at $t = 0$ and slowly approaches the final state.

A small but stable discrepancy is observed for some points around $p \sim 4\mu$ in Figs. 7 and 8. It corresponds to the numerical error which is produced due to a finite step size in the Simpson integral.

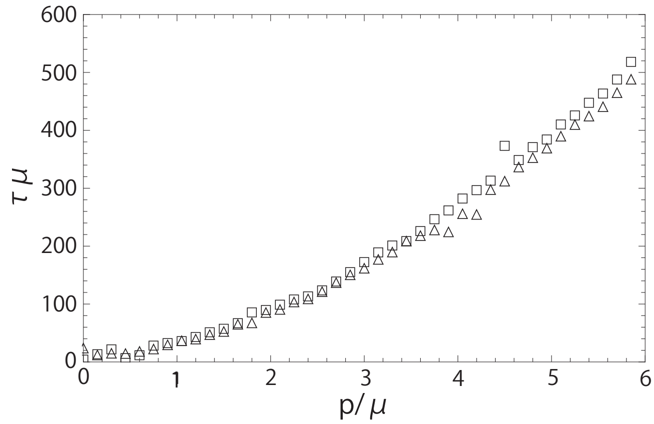

Fig. 9. Behavior of the relaxation time, τ_p. The triangle and square points show the relaxation time without and with the off-shell contribution, respectively.

In Fig. 9 both of the results are drawn in the same figure. The off-shell mode contribution does not modify the relaxation time for a lower momentum. For $p > 3\mu$ we observe a shorter relaxation time in the on-shell approximation. Since the off-shell contribution destabilizes the particle distribution near the equilibrium state, it slightly increases the relaxation time. A different contribution from the off-shell mode is expected for a system far from the equilibrium state. Because of a non-negligible numerical error it is difficult to apply our analysis to such a system. Some improvements of the numerical algorithm are necessary to extend our analysis for a general case.

§6. Conclusion

We have investigated the time evolution of the distribution function for a relativistic neutral scalar field with a $\lambda\phi^4$ interaction. The NETFD is applied to the SD equation for the scalar field propagator. Calculating the 2-loop thermal self-energy and inserting it into the SD equation, we have derived the time evolution equation for the thermal Bogoliubov parameter. The equation has the same structure with the Boltzmann equation. Therefore the Boltzmann equation is obtained from the SD equation at the equal time limit.

The Boltzmann equation consists of the collision terms which depend on the thermal Bogoliubov parameter and an oscillating coefficient with respect to the time variable. The oscillating coefficient has a peak for the on-shell case. It reduces to the delta function which shows the energy conservation between the collision particles in the on-shell approximation. Solving the obtained Boltzmann equation, we have evaluated the time evolution of the thermal Bogoliubov parameter. We suppose that the scalar field loses 20% of mass suddenly at $t = 0$. It is observed that the thermal Bogoliubov parameter approaches an equilibrium state. The relaxation time monotonically increases as a function of the momentum. In our setup the off-shell mode has only a small contribution to the particle number density. The off-shell mode

tends to suppress the relaxation near the equilibrium state and slightly increases the relaxation time for a higher momentum region.

We have derived the Boltzmann equation from the SD equation. For a non-relativistic scalar field the Boltzmann equation has been derived as the self-consistent renormalization condition based on the canonical quantization by H. Umezawa and Y. Yamanaka.[1),6)–8)] In this case it has been shown that the thermal Bogoliubov parameter which satisfies the self-consistent renormalization condition coincides with the observed particle number density.[1),22),23)] We would like to show the correspondence between the thermal Bogoliubov parameter and the particle number density in our approach. For this purpose it is necessary to construct our procedure based on the canonical quantization.

In the derivation of the time evolution equation the second derivative of the Bogoliubov parameter is canceled out at the equal time limit. Thus we obtain the Markovian equation. In the calculation of the self-energy part we impose the internal scalar propagator to be in the final equilibrium state. We should improve these assumptions to generalize the procedure. It is also interesting to apply our analysis to the relativistic Dirac field. These will be the subject of a forthcoming paper.

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Appendix A
—— diagonal elements of the SD equation ——

The relationship between the perturbed and the unperturbed propagator is given by the SD equations (3·1) and (3·2). Here we evaluate the diagonal elements of these equations. Applying the Bogoliubov matrix, $B^{-1}(n_H)$ and $B(n_H)$ to Eq. (3·1) on the left and the right sides, respectively, we obtain conditions for each diagonal element of the perturbed propagator.

$$\begin{aligned}
&D^{11}_{H,R}(t_x-t_y)+D^{22}_{H,A}(t_x-t_y)=D^{11}_{0,R}(t_x-t_y)+D^{22}_{0,A}(t_x-t_y)\\
&\quad+\int dt_{z_1}dt_{z_2}\Big(D^{11}_{0,R}(t_x-t_{z_1})+D^{22}_{0,A}(t_x-t_{z_1})\Big)\\
&\quad\times i\Sigma_R(t_{z_1}-t_{z_2})\Big(D^{11}_{H,R}(t_{z_2}-t_y)+D^{22}_{H,A}(t_{z_2}-t_y)\Big), \qquad (A\cdot1)\\
&D^{22}_{H,R}(t_x-t_y)+D^{11}_{H,A}(t_x-t_y)=D^{22}_{0,R}(t_x-t_y)+D^{11}_{H,A}(t_x-t_y)\\
&\quad+\int dt_{z_1}dt_{z_2}\Big(D^{22}_{0,R}(t_x-t_{z_1})+D^{11}_{0,A}(t_x-t_{z_1})\Big)\\
&\quad\times i\Sigma_A(t_{z_1}-t_{z_2})\Big(D^{22}_{H,R}(t_{z_2}-t_y)+D^{11}_{H,A}(t_{z_2}-t_y)\Big). \qquad (A\cdot2)
\end{aligned}$$

We suppose that the retarded and the advanced parts of these equations separate and impose conditions,

$$\begin{aligned}
&D^{11}_{H,R}(t_x - t_y) = D^{11}_{0,R}(t_x - t_y) &\text{(A·3)}\\
&+ \int dt_{z_1} dt_{z_2} \Big(D^{11}_{0,R}(t_x - t_{z_1}) + D^{22}_{0,A}(t_x - t_{z_1}) \Big) i\Sigma_R(t_{z_1} - t_{z_2}) D^{11}_{H,R}(t_{z_2} - t_y),\\
&D^{22}_{H,A}(t_x - t_y) = D^{22}_{0,A}(t_x - t_y) &\text{(A·4)}\\
&+ \int dt_{z_1} dt_{z_2} \Big(D^{11}_{0,R}(t_x - t_{z_1}) + D^{22}_{0,A}(t_x - t_{z_1}) \Big) i\Sigma_R(t_{z_1} - t_{z_2}) D^{22}_{H,A}(t_{z_2} - t_y),\\
&D^{22}_{H,R}(t_x - t_y) = D^{22}_{0,R}(t_x - t_y) &\text{(A·5)}\\
&+ \int dt_{z_1} dt_{z_2} \Big(D^{22}_{0,R}(t_x - t_{z_1}) + D^{11}_{0,A}(t_x - t_{z_1}) \Big) i\Sigma_A(t_{z_1} - t_{z_2}) D^{22}_{H,R}(t_{z_2} - t_y),\\
&D^{11}_{H,A}(t_x - t_y) = D^{11}_{0,A}(t_x - t_y) &\text{(A·6)}\\
&+ \int dt_{z_1} dt_{z_2} \Big(D^{22}_{0,R}(t_x - t_{z_1}) + D^{11}_{0,A}(t_x - t_{z_1}) \Big) i\Sigma_A(t_{z_1} - t_{z_2}) D^{11}_{H,A}(t_{z_2} - t_y).
\end{aligned}$$

From the SD equation (3·2) we find other conditions for the perturbed propagator.

$$\begin{aligned}
&D^{11}_{H,R}(t_x - t_y) + D^{22}_{H,A}(t_x - t_y) = D^{11}_{0,R}(t_x - t_y) + D^{22}_{0,A}(t_x - t_y)\\
&+ \int dt_{z_1} dt_{z_2} \Big(D^{11}_{H,R}(t_x - t_{z_1}) + D^{22}_{H,A}(t_x - t_{z_1}) \Big)\\
&\quad \times i\Sigma_R(t_{z_1} - t_{z_2}) \Big(D^{11}_{0,R}(t_{z_2} - t_y) + D^{22}_{0,A}(t_{z_2} - t_y) \Big), &\text{(A·7)}\\
&D^{22}_{H,R}(t_x - t_y) + D^{11}_{H,A}(t_x - t_y) = D^{22}_{0,R}(t_x - t_y) + D^{11}_{H,A}(t_x - t_y)\\
&+ \int dt_{z_1} dt_{z_2} \Big(D^{22}_{H,R}(t_x - t_{z_1}) + D^{11}_{H,A}(t_x - t_{z_1}) \Big)\\
&\quad \times i\Sigma_A(t_{z_1} - t_{z_2}) \Big(D^{22}_{0,R}(t_{z_2} - t_y) + D^{11}_{0,A}(t_{z_2} - t_y) \Big). &\text{(A·8)}
\end{aligned}$$

In a similar manner with Eqs. (A·3)-(A·6) we divide Eqs. (A·7) and (A·8),

$$\begin{aligned}
&D^{11}_{H,R}(t_x - t_y) = D^{11}_{0,R}(t_x - t_y) &\text{(A·9)}\\
&+ \int dt_{z_1} dt_{z_2} D^{11}_{H,R}(t_x - t_{z_1}) i\Sigma_R(t_{z_1} - t_{z_2}) \Big(D^{11}_{0,R}(t_{z_2} - t_y) + D^{22}_{0,A}(t_{z_2} - t_y) \Big),\\
&D^{22}_{H,A}(t_x - t_y) = D^{22}_{0,A}(t_x - t_y) &\text{(A·10)}\\
&+ \int dt_{z_1} dt_{z_2} D^{22}_{H,A}(t_x - t_{z_1}) i\Sigma_R(t_{z_1} - t_{z_2}) \Big(D^{11}_{0,R}(t_{z_2} - t_y) + D^{22}_{0,A}(t_{z_2} - t_y) \Big),\\
&D^{22}_{H,R}(t_x - t_y) = D^{22}_{0,R}(t_x - t_y) &\text{(A·11)}\\
&+ \int dt_{z_1} dt_{z_2} D^{22}_{H,R}(t_x - t_{z_1}) i\Sigma_A(t_{z_1} - t_{z_2}) \Big(D^{22}_{0,R}(t_{z_2} - t_y) + D^{11}_{0,A}(t_{z_2} - t_y) \Big),\\
&D^{11}_{H,A}(t_x - t_y) = D^{11}_{0,A}(t_x - t_y) &\text{(A·12)}\\
&+ \int dt_{z_1} dt_{z_2} D^{11}_{H,A}(t_x - t_{z_1}) i\Sigma_A(t_{z_1} - t_{z_2}) \Big(D^{22}_{0,R}(t_{z_2} - t_y) + D^{11}_{0,A}(t_{z_2} - t_y) \Big).
\end{aligned}$$

The self-energy, $\Sigma_{R,A}$ has retarded and advanced time dependence, $\Sigma_R(t-t')=0$ for $t<t'$ and $\Sigma_A(t-t')=0$ for $t>t'$. The time dependence for D_R and D_A is given in Eqs. (2·20)-(2·23). Thus the quantum corrections for the perturbed propagators (A·3) -(A·6) and (A·9)-(A·12) disappear at the equal time limit. As a result the perturbed propagators at the equal time limit are derived from the unperturbed propagators (2·20)-(2·23).

$$\lim_{t_x\to t_y} D^{11}_{H,R}(t_x-t_y)=\lim_{t_x\to t_y} D^{11}_{H,A}(t_x-t_y)=\frac{1}{4\omega_p}, \tag{A·13}$$

$$\lim_{t_x\to t_y} D^{22}_{H,R}(t_x-t_y)=\lim_{t_x\to t_y} D^{11}_{H,A}(t_x-t_y)=-\frac{1}{4\omega_p}. \tag{A·14}$$

Substituting the expressions (2·20)-(2·23) for the unperturbed propagators, we obtain the first derivative of the perturbed propagators at the equal time limit,

$$\lim_{t_x\to t_y} \partial_{t_x} D^{22}_{H,R}(t_x-t_y)=\lim_{t_x\to t_y} \partial_{t_y} D^{22}_{H,A}(t_x-t_y)=\frac{i}{2}+\lim_{t_x\to t_y}\delta(t_x-t_y), \tag{A·15}$$

$$\lim_{t_x\to t_y} \partial_{t_x} D^{11}_{H,A}(t_x-t_y)=\lim_{t_x\to t_y} \partial_{t_y} D^{11}_{H,R}(t_x-t_y)=\frac{i}{2}-\lim_{t_x\to t_y}\delta(t_x-t_y). \tag{A·16}$$

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